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Fluctuations in Derrida's Random Energy and Generalized Random Energy Models

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The fluctuations of the finite-size corrections to the free energy per site of the random energy model (REM) and the generalized random energy model (GREM) are investigated. Almost sure behavior for the corrections of order $(\log N)/N$ is given. We also prove convergence in distribution for the corrections of order $1/N$.

KEY WORDS: Random energy; random variables; finite-size corrections; Poisson point process.

1. INTRODUCTION

Random energy models (REMs) were introduced by Derrida[(1)] as simple and solvable models for spin glasses. Spin glasses are disordered magnetic systems and a mean field description of such system is given by the Sherrington–Kirkpatrick (SK) model.[(2)] For a recent review we refer to the Mezard *et al.*[(3)] and references included there.

In their studies of the SK model, Mezard *et al.*[(4)] discuss the role of the fluctuations of order $1/N$ of the free energy to define the weights of the pure states of the model. They also sketch a self-consistent approach that, under a suitable hypothesis for these weights, should allow one to get the replica symmetry-breaking solution of the SK model without introducing the replicas.[(5)]

From this analysis, a picture emerges where the study of fluctuating quantities plays a major role. An example is the distribution of overlap-

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pings for the SK model.[6] Let us also mention the work of Derrida and Toulouse[7] for the REM and de Dominicis and Hilhorst[8] for the generalized REM (GREM).

In this paper we study the finite-size correction to the free energy per site for the REM and prove that if $\beta > \beta_c$, fluctuations exists which are sample dependent. This clarifies a point of the work of Olivieri and Picco.[10] Moreover, we find a scaling which allows us to prove that in the limit $N \to \infty$ the Boltzman factors are realizations of a Poisson point process as stated by Ruelle.[9]

The case of the GREM is also considered.

In Section 2 we state the results in a precise form, and in Section 3 we give the proofs.

2. DEFINITIONS AND RESULTS

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and assume that there exists a family $\{X_i\}$, $i = 1, 2,...,$ of independent normalized Gaussian random variables that are defined on $(\Omega, \Sigma, \mathbb{P})$. Let β be a positive real number; we define the random variables

$$Z_N(\beta) = \sum_{i=1}^{2^N} \exp \beta N^{1/2} X_i \tag{2.1}$$

and

$$F_N(\beta) = \frac{1}{N} \log Z_N(\beta) \tag{2.2}$$

where $N = 1, 2,....$ Let us define also $\beta_c = (2 \log 2)^{1/2}$ and

$$F(\beta) = \begin{cases} \beta^2/2 + \beta_c^2/2 & \text{if } \ 0 \leqslant \beta \leqslant \beta_c \\ \beta\beta_c & \text{if } \ \beta \geqslant \beta_c \end{cases} \tag{2.3}$$

It is proved in ref. 10 that $\forall \beta > 0$, $F_N(\beta)$ converge almost everywhere to $F(\beta)$. Moreover, it follows from Proposition 5 of ref. 10 that if $\beta \leqslant \beta_c$, then

$$\lim_{N \to \infty} Z_N(\beta) e^{-NF(\beta)} = 1 \quad \text{almost everywhere} \tag{2.4}$$

Let us define for $\beta > 0$

$$\mathscr{H}_N(\beta) = \frac{\beta_c}{\beta} \frac{\log[Z_N e^{-NF(\beta)}]}{\log N} \tag{2.5}$$

$\mathscr{H}_N$ is the finite-size correction of order $(\log N)/N$ to the free energy.

The following theorem states our results on the fluctuation of the finite-size corrections of order $(\log N)/N$ to the free energy per site.

Theorem 1. If $0<\beta<\beta_c$,

$$\lim \mathscr{H}_N(\beta)=0 \quad \text{almost everywhere} \tag{2.6}$$

If $\beta \geqslant \beta_c$,

$$\limsup \mathscr{H}_N(\beta)=1/2 \quad \text{almost everywhere} \tag{2.7}$$

and

$$\liminf \mathscr{H}_N(\beta)=-1/2 \quad \text{almost everywhere} \tag{2.8}$$

Remark. As a consequence of Propositions 2 and 4 of ref. 10 we get, if $\beta>\beta_c$, $\lim_{N\to\infty} \mathscr{H}_N(\beta)=-1/2$ in probability; in fact, the probability that $|\mathscr{H}_N(\beta)+1/2|\leqslant \varepsilon$ goes to one like $1-\text{cte}/(\log N)^{1+\varepsilon}$.

This remark suggests the natural scale to study the Boltzmann factors $\exp \beta N^{1/2} X_i$ when $\beta>\beta_c$. We define

$$\xi_i^N=\exp\left(\beta N^{1/2}X_i-\beta\beta_c N+\frac{1}{2}\frac{\beta}{\beta_c}\log N\right) \tag{2.9}$$

and

$$\mathscr{N}_t^{(N)}=\sum_{i=1}^{2^N} \mathbb{1}_{\geqslant \exp(\beta t/\beta_c-\beta/\beta_c \log \beta_c(2\pi)^{1/2})}(\xi_i^N)/2 \tag{2.10}$$

where

$$\mathbb{1}_{\geqslant\alpha}(x)=\begin{cases}1 & \text{if } x\geqslant\alpha\\ 0 & \text{if } x<\alpha\end{cases}$$

$\mathscr{N}_t^{(N)}$ is related to the counting measure for the point process defined by $(\xi_i^N)_{i=1}^{2^N}$. The following theorem gives the sample-dependent fluctuations of the Boltzmann factors and it is equivalent to the convergence in law of the point process (ξ_i^N). Definitions and basic results for point proccesses can be found in Neveu.[13]

Theorem 2. The $\mathscr{N}_\cdot^{(N)}$ converge in law to the counting measure of a Poisson point process with intensity $\exp(-t)\,\lambda(dt)$.

Remark. By a simple change of variable this mean that ξ_i^N converge to a Poisson point process with intensity

$$\frac{\beta_c}{\beta}\frac{\lambda(dx)}{X^{1+\beta_c/\beta}}\mathbb{1}_{\geqslant 0}(x)$$

where $\lambda(dx)$ is the Lebesgue measure.

This is the assumed distribution for the "Boltzmann factor" in the paper of Ruelle[9] and gives the interpretation of the parameter: $\rho = \beta_c$. This distribution was also introduced by Mezard *et al.*[4] and De Dominicis and Hilhorst[8] on the basis of heuristic arguments. After completion of this work we discovered that Theorem 2 is not new and appears in Leadbetter *et al.*[12] (see Theorem 1.5.3 and Theorem 2.11). Since our proof is different from theirs and we need Lemma 4 for proving Theorem 3, we keep it for convenience.

In the case of the GREM we have to consider cascades[9] instead of Poisson point processes. We will consider for simplicity the case of the GREM with two hierarchies; the general case is an easy generalization as long as the number of hierarchies n is smaller than $N/(\log N)^{1+\varepsilon}$; see Capocaccia *et al.*,[11] where such a restriction is discussed. Let us now define the GREM with two hierarchies. Let $(\alpha_i)_{i=1,2}$ and $(a_i)_{i=1,2}$ be real positive numbers such that $\log \alpha_1 + \log \alpha_2 = \log 2$, $a_1 + a_2 = 1$, and let $(\Omega, \Sigma, \mathbb{P})$ be a probability space such that for any $N \in \mathbb{N}^*$ there exists two families $\{\varepsilon_{k_1}\}$, $k_1 = 1,..., \alpha_1^N$, and $\{\varepsilon_{k_1 k_2}\}$, $k_1 = 1,..., \alpha_1^N$, $k_2 = 1,..., \alpha_2^N$, of independent normalized Gaussian random variables that are defined on $(\Omega, \Sigma, \mathbb{P})$. Let us define, as previously, renormalized Boltzmann factors:

$$\xi_{k_1} = \exp\left\{\beta(Na_1)^{1/2}\varepsilon_{k_1} - \beta a_1^{1/2}(2\log\alpha_1)^{1/2}N + \frac{1}{2}\frac{\beta a_1^{1/2}\log N}{(2\log\alpha_1)^{1/2}}\right\} \tag{2.11}$$

$$\xi_{k_1 k_2} = \exp\left\{\beta(Na_2)^{1/2}\varepsilon_{k_1 k_2} - \beta a_2^{1/2}(2\log\alpha_2)^{1/2}N + \frac{1}{2}\frac{\beta a_2^{1/2}\log N}{(2\log\alpha_2)^{1/2}}\right\} \tag{2.12}$$

$$\tilde{G}_N^i(t) = \exp\left\{\frac{\beta a_i^{1/2} t}{(2\log\alpha_i)^{1/2}} - \beta a_i^{1/2}\frac{\log(4\pi\log\alpha_i)^{1/2}}{(2\log\alpha_i)^{1/2}}\right\} \tag{2.13}$$

An illuminating way to consider cascades[9] is to consider the equivalent of the counting measure $\mathcal{N}_t^{(N)}$, (2.9), namely the two-parameter process

$$\mathcal{N}_{t_1,t_2}^{(N)} = \sum_{k_1=1}^{\alpha_1^N} \mathbb{1}_{\geqslant \tilde{G}_1^N(t_2)}(\xi_{k_1}) \sum_{k_1=1}^{\alpha_2^N} \mathbb{1}_{\geqslant \tilde{G}_N^2(t_2)}(\xi_{k_1,k_2}) \tag{2.14}$$

If Δ is a rectangle $[t, t'] \times [s, s']$, $t < t'$, $s < s'$, we define

$$\mathcal{N}_\Delta^{(N)} = \mathcal{N}_{t,s}^{(N)} + \mathcal{N}_{t',s'}^{(N)} - \mathcal{N}_{t,s'}^{(N)} - \mathcal{N}_{t',s}^{(N)}$$

$\mathcal{N}_\Delta^{(N)}$ counts the number of points which fall in Δ. Let us remark that without the rescaling of the Boltzmann factor Capocaccia *et al.*[11] proved a strong law of large numbers for similar quantities in their proof of the existence of the infinite-volume free energy for the GREM.

The following theorem is our main result for the GREM.

Theorem 3. The two-parameter stochastic process $\mathcal{N}_{t_1 t_2}^{(N)}$ converge in law to a process $\mathcal{N}_{t_1 t_2}$ such that for any finite family of rectangles $(\Delta_i)_{i=1}^N \equiv ([t_i, t_{i+1}] \times [S_i, S_{i+1}]$ with $t_i \leqslant t_{i+1} \leqslant t_{i+2}$, $S_i \leqslant S_{i+1} \leqslant S_{i+2} \ldots)$ the random variables $\mathcal{N}_{\Delta_1}, \ldots, \mathcal{N}_{\Delta_n}$ are independent and

$$\begin{aligned}
&\mathbb{E}(\exp(-S\mathcal{N}_{t_1 t_2})) \\
&\quad = \exp[+\exp(-t_1)(\exp\{[+\exp(-t_2)][\exp(-S)-1]\}-1)] \qquad (2.15)
\end{aligned}$$

Remark. We emphasize that for fixed t_1 the process $\mathcal{N}_{t_1, t_2}$ is not with independent increments with respect to t_2, that is, if $t_2' > t_2$, $\mathcal{N}_{t_1 t_2} - \mathcal{N}_{t_1 t_2'}$ and $\mathcal{N}_{t_1 t_2'}$ are not independent random variables, which will be the case for the counting measure of a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}^+$.

3. PROOFS

Proof of Theorem 1. As a direct consequence of Proposition 2 of ref. 10 we get

$$\operatorname{Prob}\left(Z_N \exp[-F(\beta)N] \geqslant \exp -\beta \frac{\log N}{\beta_c}\left(\frac{1}{2}+\varepsilon\right)\right.$$
$$\left.\text{for all but finite number of } N\right) = 1 \qquad (3.1)$$

Moreover, using the fact that if A_n is a sequence of events such that $\lim P(A_n) = 1$ and if we define

$$\{A_n \text{ infinitely often}\} = \{A_n \text{ i.o.}\} = \bigcap_{n \geqslant 1} \bigcup_{k=n}^{\infty} \{A_k\}$$

we get

$$\mathbb{P}(A_n \text{ i.o.}) = \lim_{n \to \infty} \mathbb{P}\left(\bigcup_{k=n}^{\infty} A_k\right) \geqslant \lim_{n \to \infty} \mathbb{P}(A_n) = 1$$

and since by Proposition 4 of ref. 10 we have

$$\lim_{n \to \infty} \mathbb{P}\left(Z_N \exp[-NF(\beta)] \leqslant \exp -\frac{\beta}{\beta_c}[\log N]\left(\frac{1}{2}-\varepsilon\right)\right) = 1 \qquad (3.2)$$

we get

$$\mathbb{P}\left(Z_N \exp[-NF(\beta)] \leqslant \exp -\frac{\beta}{\beta_c}[\log N]\left(\frac{1}{2}-\varepsilon\right) \text{i.o.}\right) = 1 \qquad (3.3)$$

Therefore, using (3.1) and (3.3), we get $\liminf \mathcal{N}_N(\beta) = -1/2$.

Using Proposition 2 of ref. 10, we get

$$\mathbb{P}\left(Z_N \exp[-F(\beta)N] \leqslant \exp + \beta \frac{\log N}{\beta_c}\left(\frac{1}{2}+\varepsilon\right)\right.$$
$$\left.\text{for all but a finite number of } N\right) = 1 \tag{3.4}$$

This implies $\limsup \mathscr{H}_N(\beta) \leqslant 1/2$ almost surely.

In order to prove that $\limsup \mathscr{H}_N(\beta) = 1/2$, since

$$Z_N \exp[-F(\beta)N] \geqslant \exp\{\beta N^{1/2}(\underset{i=1-2^N}{\operatorname{Max}} X_i - \beta_c N^{1/2})\}$$

it is sufficient to prove

$$\mathbb{P}\left(\left\{\underset{i=1-2^N}{\operatorname{Max}} X_i \geqslant \beta_c N^{1/2} + \left(\frac{1}{2}-\varepsilon\right)\frac{\log N}{\beta_c N^{1/2}}\right\} \text{ i.o.}\right) = 1 \tag{3.5}$$

Let us write

$$B_N = \left\{\underset{i=1-2^N}{\operatorname{Max}} X_i \geqslant \beta_c N^{1/2} + \left(\frac{1}{2}-\varepsilon\right)\frac{\log N}{\beta_c N^{1/2}}\right\}$$

and remark that

$$B_N = \bigcup_{i=1}^{2^N} A_{i,2^N}$$

where

$$A_{i,2^N} = \left\{X_i \geqslant \beta_c N^{1/2} + \left(\frac{1}{2}-\varepsilon\right)\frac{\log N}{\beta_c N^{1/2}}\right\} \tag{3.6}$$

In order to prove $\mathbb{P}(\limsup B_N) = 1$ it is sufficient to prove

$$\lim_{m\to\infty} \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{k=m}^{n} B_k^c\right) = 0 \tag{3.7}$$

or similarly

$$\lim_{m\to\infty} \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{k=m}^{n}\left(\bigcap_{i=1}^{2^k} A_{i,2^k}^c\right)\right) = 0 \tag{3.8}$$

It is not too difficult to see that

$$\bigcap_{k=m}^{n} \bigcap_{i=1}^{2^k} A_{i,2^k}^c = \left[\bigcap_{i=1}^{2^m}\left(\bigcap_{k=m}^{n} A_{i,2^k}^c\right)\right] \cap \left[\bigcap_{i=2^m+1}^{2^{m+1}}\left(\bigcap_{k=m+1}^{n} A_{i,2^k}^c\right)\right] \cap$$
$$\cdots \cap \left[\bigcap_{i=2^{m+s}+1}^{2^{m+s+1}}\left(\bigcap_{k=m+1+s}^{n} A_{i,2^k}^c\right)\right] \cdots \cap \left(\bigcap_{i=2^{n-1}+1}^{2^n} A_{i,2^n}^c\right) \tag{3.9}$$

Now, since the function

$$f(x)=(2x\log 2)^{1/2}+\frac{1}{2}-\varepsilon\frac{\log X}{\beta_c x^{1/2}}$$

is strictly increasing if X is large enough, we get:
if m is large enough

$$\bigcap_{k=m+s+1}^{n} A_{i,2^k}^c = A_{i,2^{m+s+1}}^c \tag{3.10}$$

Therefore, using independence, (3.9), and (3.10), we get

$$\begin{aligned}\mathbb{P}&\left(\bigcap_{k=m}^{n}\bigcap_{i=1}^{2^k} A_{i,2^k}^c\right)\\ &=\left[\prod_{i=1}^{2^m}\mathbb{P}(A_{i,2^m}^c)\right]\left[\prod_{i=2^m+1}^{2^{m+1}}\mathbb{P}(A_{i,2^{m+1}}^c)\right]'\\ &\quad\times\left[\prod_{i=2^{m+s}+1}^{2^{m+s+1}}\mathbb{P}(A_{i,2^{m+s+1}}^c)\right]\left[\prod_{i=2^{n-1}+1}^{2^n}\mathbb{P}(A_{i,2^n}^c)\right]\end{aligned} \tag{3.11}$$

Using now

$$\mathbb{P}(A_{i,2^m}^c)=1-\mathbb{P}(A_{i,2^m})\leqslant\exp[-\mathbb{P}(A_{i,2^m})] \tag{3.12}$$

and writing

$$P(A_{i,2^m})=c(m)=\text{Prob}\left(X\geqslant m^{1/2}\beta_c+\left(\frac{1}{2}-\varepsilon\right)\frac{\log m}{m^{1/2}\beta_c}\right) \tag{3.13}$$

we get

$$P\left(\bigcap_{k=m}^{n}\bigcap_{i=1}^{2^k} A_{i,2^k}^c\right)\leqslant\exp\left\{-\left[2^m c(m)+\sum_{k=m+1}^{n}2^{k-1}c(k)\right]\right\} \tag{3.14}$$

Since

$$\int_x^{+\infty}\exp-\frac{u^2}{2}\,du>\frac{x}{1+x^2}\exp-\frac{x^2}{2} \tag{3.15}$$

it is not difficult to check that there exists a constant C such that if m is large enough, $\forall K>m$, $2^{k-1}C(k)\geqslant C/K^{1-\varepsilon}$. Therefore

$$\lim_{n\to\infty}\sum_{k=m+1}^{n}2^{k-1}c(k)=+\infty$$

from which, using (3.14), we get (3.8) and this ends the proof of Theorem 1.

Proof of Theorem 2. The proof of Theorem 2 will be a direct consequence of the two following lemmata:

Lemma 4. Let X be a Gaussian normalized random variable, and

$$g_N(t) = N^{1/2}\beta_c - \frac{1}{\beta_c}\frac{\log \beta_c(2\pi N)^{1/2}}{N^{1/2}} + \frac{t}{\beta_c N^{1/2}} \tag{3.16}$$

where $\beta_c = (2\log\alpha)^{1/2}$ and α is a real number >1.

Then

$$\lim_{n\to\infty} \alpha^N \mathbb{P}(x \geqslant g_N(t)) = \exp(-t) \tag{3.17}$$

Lemma 5. Let $0 < t_p < t_{p-1} < \cdots < t_1 < t_0 = +\infty$ be a sequence of real, positive numbers and $\{S_i\}_i^p = 1$ a sequence of real numbers. Then

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left(\exp\left[-\sum_{k=1}^{p} S_k(\mathcal{N}_{t_k}^{(N)} - \mathcal{N}_{t_{k-1}}^{(N)})\right]\right)\\
&\quad = \prod_{k=1}^{p} \lim_{n\to\infty} \mathbb{E}\left(\exp\left[-\sum_{k=1}^{p} S_k(\mathcal{N}_{t_k}^{(N)} - \mathcal{N}_{t_{k-1}}^{(N)})\right]\right)\\
&\quad = \exp\sum_{k=1}^{p} [\exp(S_k)-1][\exp(-t_{k-1}) - \exp(-t_k)]
\end{aligned} \tag{3.18}$$

Proof of Lemma 4. Using the fact that for a Gaussian normalized random variable

$$P(X \geqslant t) \leqslant (2\pi t)^{-1/2}\exp(-t^2/2) \tag{3.19}$$

it is not difficult to check that if t^2/N and $(\log N)^2/N$ are small enough, then, for some constant C_1,

$$\left|\frac{\alpha^N \exp(-g_N^2/2)}{(2\pi)^{1/2} g_N(t)} - \exp(-t)\right| \leqslant \left[\frac{2t+t^2}{\beta_c^2 N} + C_1\frac{(\log N)^2}{N}\right]\exp(-t) \tag{3.20}$$

On the other hand, for a Gaussian normalized random variable

$$P(X \geqslant t) \geqslant \frac{\exp(-t^2/2)}{(2\pi)^{1/2}t}\left(1 - \frac{1}{t^2}\right) \tag{3.21}$$

and if N is large enough, there exist a constant C_2 such that $g_N^2(t) \geqslant N/C_2$. Therefore, using (3.20), we obtain

$$\begin{aligned}
&\frac{\alpha^N \exp[-g_N^2(t)/2]}{(2\pi)^{1/2} g_N(t)}\left(1 - \frac{1}{g_N^2(t)}\right)\\
&\quad \geqslant \left(1 - \frac{C_2}{N}\right)\left\{1 - \left[\frac{2t+t^2}{\beta_c^2 N} + C_1\frac{(\log N)^2}{N}\right]\right\}\exp(-t)
\end{aligned} \tag{3.22}$$

From (3.20) and (3.22) we get the result.

Proof of Lemma 5. Since

$$\mathbb{E}\left(\prod_{k=1}^{p} \exp\left[-S_k(\mathcal{N}_{t_k}^{(N)} - \mathcal{N}_{t_{k-1}}^{(N)})\right]\right)$$

$$= \mathbb{E}\left(\prod_{i=1}^{\alpha^n} \left\{\prod_{k=1}^{p} \exp[-S_k(\mathbb{1}_{\geqslant g_N(t_k)}(X_i) - \mathbb{1}_{\geqslant g_N(t_{k-1})}(X_i))]\right\}\right) \tag{3.23}$$

and the random variables which appear in the bracket $\{\cdot\}$ are independent for different indices i, writing

$$-\mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X_i) = \mathbb{1}_{\geqslant g_N(t_k)}(X_i) - \mathbb{1}_{\geqslant g_N(t_{k-1})}(X_i)$$

we get

$$(3.23) = \left[\mathbb{E}\left(\prod_{k=1}^{p} \exp S_k \mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X)\right)\right]^{\alpha^N} \tag{3.24}$$

Since

$$\exp[\alpha \mathbb{1}_A(X)] = 1 + (e^\alpha - 1)\mathbb{1}_A(X)$$

we get

$$\prod_{k=1}^{p} \exp[S_k \mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X)]$$

$$= \prod_{k=1}^{p} \{1 + [\exp(S_k) - 1]\, \mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X)\} \tag{3.25}$$

Using the fact that the intervals

$$[g_N(t_{k_i-1}), g_N(t_{k_i})[\qquad \text{and} \qquad [g_N(t_{k_j-1}), g_N(t_{k_j})[$$

are disjoint if $i \neq j$, if we expand the previous product, the only nonzero terms which remain are

$$1 + \sum_{k=1}^{p} (e^{S_k} - 1)\, \mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X) \tag{3.26}$$

Therefore (3.23) is equal to

$$\exp\left\{\alpha^N \log\left(1 + \sum_{k=1}^{p} [\exp(S_k) - 1]\, \mathbb{E}[\mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X)]\right)\right\} \tag{3.27}$$

Since if $|x|<1$

$$|\log(1+X)-X| \leqslant \frac{X^2}{2}\frac{1}{1-|X|} \tag{3.28}$$

and

$$\lim \alpha^N \mathbb{E}(\mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X)) = \exp(-t_{k-1}) - \exp(-t_k)$$

then by Lemma 4, for fixed $p \geqslant 1$ and $\{S_k\}$, $\{t_k\}$, $k=1,..., p$, (3.28) implies that there exists a constant C_p such that, if N is large enough, then

$$\begin{aligned} \Bigg| \log\left(1+\sum_{k=1}^{p}(e^{S_k}-1)\,\mathbb{E}(\mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X)\right) \\ - \sum_{k=1}^{p}(e^{S_k}-1)\,\mathbb{E}(\mathbb{1}_{g_N(t_{k-1}),\, g_N(t_k)}(X)\Bigg| \\ \leqslant \frac{C_p \mathscr{F}^2}{\alpha^{2N}(1-C_p\,\mathscr{F}/\alpha^N)} \end{aligned} \tag{3.29}$$

where

$$\mathscr{F} = \sum_{k=1}^{p}(e^{S_k}-1)(e^{-t_{k-1}}-e^{-t_k})$$

Therefore we get

$$\begin{aligned} \lim_{N\to\infty} \mathbb{E}\left(\prod_{k=1}^{p}\exp[-S_k(\mathscr{N}_{t_k}^{(N)}-\mathscr{N}_{t_{k-1}}^{(N)})]\right) \\ = \exp\sum_{k=1}^{p}[\exp(S_k)-1][\exp(-t_{k-1})-\exp(-t_k)] \end{aligned} \tag{3.30}$$

concluding the proof of Lemma 5.

Proof of Theorem 3. Let us first prove formula (2.16). It is not difficult to check that

$$\begin{aligned} &\mathbb{E}(\exp[-S\mathscr{N}_{t_1t_2}^{(N)}]) \\ &= \mathbb{E}\left(\prod_{k=1}^{\alpha_1^N}\exp\left[-S\mathbb{1}_{\geqslant \tilde{g}_N^1(t_1)}(\xi_{k_1})\sum_{k_2=1}^{\alpha_2^N}\mathbb{1}_{\geqslant \tilde{g}_N^2(t_2)}(\xi_{k_1,k_2})\right]\right) \end{aligned} \tag{3.31}$$

We remark that the random variables

$$\eta_{k_1} = \exp\left[-S\mathbb{1}_{\geqslant \tilde{g}_N^1(t_1)}(\xi_{k_1})\sum_{k_2=1}^{\alpha_2^N}\mathbb{1}_{\geqslant \tilde{g}_N^2(t_2)}(\xi_{k_1,k_2})\right], \qquad k_1 = 1,..., \alpha_1^N \tag{3.32}$$

are independent and identically distributed. Therefore

$$\mathbb{E}(\exp[-S\mathcal{N}_{t_1t_2}^{(N)}])$$
$$=\left[\mathbb{E}\exp\left\{-S\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\sum_{k_2=1}^{\alpha_2^N}\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\varepsilon_{k_2})\right)\right]^{\alpha_1^N} \tag{3.33}$$

where ε and $\varepsilon_{k_2}, k_2=1,..., \alpha_2^N$, are independent, normalized, Gaussian random variables and

$$g_N^{(i)}(t)=(N2\log\alpha_i)^{1/2}-\frac{\log(4\pi N\log\alpha_i)^{1/2}}{(2N\log\alpha_i)^{1/2}}+\frac{t}{(2N\log\alpha_i)^{1/2}}$$

Now for *fixed* ε the random variables

$$\exp\left[-S\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\sum_{k_2=1}^{\alpha_2^N}\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\varepsilon_{k_2})\right],\qquad k_2=1,..., \alpha_2^N$$

are independent, identically distributed; therefore, if ε and η are independent, normalized, Gaussian random variables, (3.33) is equal to

$$[\mathbb{E}_\varepsilon\{\mathbb{E}_\eta\exp[-S\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta)]\}^{\alpha_2^N} \tag{3.34}$$

Using

$$\exp[-s\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta)]$$
$$=1+(e^{-s}-1)\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta) \tag{3.35}$$

and integrating with respect to η, we get

$$(3.34)=[\mathbb{E}_\varepsilon\{1+(e^{-s}-1)\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\mathbb{E}_\eta\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta)\}^{\alpha_1^N} \tag{3.36}$$

Since

$$(1-x)^{\alpha_2^N}\leqslant\exp(-x\alpha_2^N)\qquad\text{for}\quad x>0 \tag{3.37}$$

we get

$$(3.36)\leqslant[\mathbb{E}_\varepsilon\exp\{\alpha_2^N\mathbb{E}_\eta(\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta))[\exp(-S)-1]\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\}]^{\alpha_1^N} \tag{3.38}$$

Writing

$$\alpha_2^N\mathbb{E}_\eta(\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta)(e^{-s}-1)=\hat{S}_N$$
$$\hat{S}=e^{-t_2}(e^{-S}-1)$$

we have that Lemma 4 implies that $\lim_{N\to\infty}\hat{S}_N=\hat{S}$.

Using

$$\mathbb{E}_\varepsilon(\exp[\hat{S}_N \mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)])$$
$$= 1 + [\exp(+\hat{S}_N) - 1]\, \mathbb{E}(\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)) \tag{3.39}$$

and (3.37) we get

$$(3.38) \leqslant \exp + \alpha_1^N \mathbb{E}_\varepsilon(\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)) \exp[(+\hat{S}_N) - 1] \tag{3.40}$$

Again by Lemma 4

$$\overline{\lim}\, \mathbb{E}(\exp(-S\mathcal{N}_{t_1 t_2}^{(N)}))$$
$$\leqslant \exp[+\exp(-t_1)(\exp\{+\exp(-t_2)[\exp(-S) - 1]\} - 1)] \tag{3.41}$$

Now using (3.28), we get

$$\{1 + [\exp(-S) - 1]\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\, \mathbb{E}_\eta(\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta))\}^{\alpha_2^N}$$
$$\geqslant \exp\{[\exp(-S) - 1]\, \alpha_2^N \mathbb{E}_\eta(\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta))\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)\}$$
$$\times \left[1 - [\exp(-S) - 1]\frac{\mathbb{E}_\eta(\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta))}{2[1 - \mathbb{E}_\eta(\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta))\,|\exp(-S) - 1|]}\right] \tag{3.42}$$

Calling the term into the large bracket $[\cdot] = q_N(s, t_2)$ by Lemma 4, it is easy to check that

$$-\tilde{S}_N = (e^{-s} - 1)\, \alpha_2^N \mathbb{E}_\eta(\mathbb{1}_{\geqslant g_N^{(2)}(t_2)}(\eta))\, q_N(s, t_2) \tag{3.43}$$

goes to $(e^{-s} - 1)e^{-t_2} = \hat{S}$ when N goes to infinity. Therefore

$$(3.36) \geqslant \{\mathbb{E}_\varepsilon(\exp[-\tilde{S}_N \mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon)])\}^{\alpha_1^N}$$
$$= \exp[\alpha_1^N \log\{1 + [\exp(-\tilde{S}_N) - 1]\, \mathbb{E}(\mathbb{1}_{\geqslant g_N^{(1)}(t_1)}(\varepsilon))\}] \tag{3.44}$$

and by a similar argument as at the end of the proof of Lemma 5 we get

$$\lim_{N \to \infty} \mathbb{E}(\exp[-S\mathcal{N}_{t_1 t_2}^{(N)}])$$
$$\geqslant \exp[(\exp - t_1)(\exp\{[+\exp(-t_2)][\exp(-S) - 1]\} - 1)] \tag{3.45}$$

Collecting (3.41) and (3.45), we get (2.14).

It remains to prove that $\mathcal{N}_{\Delta_1},..., \mathcal{N}_{\Delta_n}$ are independent random variables; to do so it is sufficient to show that for all $p \geqslant 1$, for all positive real numbers r_j, $j = 1,..., p$ and for any family of rectangles $\Delta_j = [t_j, t_{j+1}[\times [s_j, s_{j+1}[$, with

$$t_j < t_{j+1} < t_{j+2}, \qquad \forall j = 1,..., p-2$$
$$s_j < s_{j+1} < s_{j+2}, \qquad \forall j = 1,..., p-2$$

one has

$$\lim_{N\to\infty} \mathbb{E}\left(\exp\left[-\sum_{j=1}^{p} r_j \mathcal{N}_{\Delta_j}^{(N)}\right]\right) = \prod_{j=1}^{p} \lim_{N\to\infty} \mathbb{E}(\exp[-r_j \mathcal{N}_{\Delta_j}^{(N)}]) \quad (3.46)$$

Since it is not difficult to check that

$$\mathcal{N}_{\Delta_j}^{(N)} = \sum_{k_1=1}^{\alpha_1^N} \mathbb{1}_{g_N^{(1)}(t_j),\, g_N^{(1)}(t_{j+1})}(\varepsilon_{k_1}) \sum_{k_2=1}^{\alpha_2^N} \mathbb{1}_{\geqslant g_N^{(2)}(s_j),\, g_N^{(2)}(S_{j+1})}(\varepsilon_{k_1k_2})$$

we get

$$\mathbb{E}\left(\prod_{j=1}^{p} \exp[-r_j \mathcal{N}_{\Delta_j}^{(N)}]\right) = \mathbb{E}\left(\prod_{k_1=1}^{\alpha_1^N} \prod_{j=1}^{p} \exp[-r_j \mathbb{1}_{\Delta_j^1}(\varepsilon_{k_1})] \sum_{k_2=1}^{\alpha_2^N} \mathbb{1}_{\Delta_j^2}(\varepsilon_{k_1k_2})\right) \tag{3.47}$$

where

$$\Delta_j^1 = [g_N^{(1)}(t_j),\, g_N^{(1)}(t_j+1)[$$

$$\Delta_j^2 = [g_N^{(2)}(s_j),\, g_N^{(2)}(s_j+1)[$$

$$\mathbb{1}_{\Delta_j^l}(x) = \begin{cases} 1, & \text{if } \; x \in \Delta_j^l \\ 0, & \text{otherwise} \end{cases}$$

Using the independence, (3.47) is equal to

$$\left\{\mathbb{E}\left(\prod_{j=1}^{p} \exp[-r_j \mathbb{1}_{\Delta_j^1}(\varepsilon)] \sum_{k_1=1}^{\alpha_2^N} \mathbb{1}_{\Delta_j^2}(\varepsilon_{k_2})\right)\right\}^{\alpha_1^N} \tag{3.48}$$

where ε and $\varepsilon_{k_1}, k_2 = 1,..., \alpha_2^N$, are independent, normalized, Gaussian Random variables.

Repeating what we did in the proof of Lemma 5, we get

$$(3.48) = \left[\mathbb{E}_\varepsilon\left(\left\{\mathbb{E}_\eta\left(\prod_{j=1}^{p} \exp[-r_j \mathbb{1}_{\Delta_j^1}(\varepsilon)\, \mathbb{1}_{\Delta_j^1}(\eta)]\right)\right\}^{\alpha_2^N}\right)\right]^{\alpha_1^N} \tag{3.49}$$

where ε and η are independent, normalized, Gaussian random variables. Using

$$\exp[-r_j \mathbb{1}_{\Delta_j^1}(\varepsilon)\, \mathbb{1}_{\Delta_j^2}(\eta)] = 1 + [\exp(-r_j) - 1]\, \mathbb{1}_{\Delta_j^1}(\varepsilon)\, \mathbb{1}_{\Delta_j^2}(\eta)$$

and the fact that Δ_j^2 and Δ_k^2 are disjoint if $j \neq k$, we expand the product $\prod_{j=1}^{p}$ in (3.49) and get

$$\left\{\mathbb{E}_\eta\left(\prod_{j=1}^{p} \exp[-r_j \mathbb{1}_{\Delta_j^1}(\varepsilon)\, \mathbb{1}_{\Delta_j^2}(\eta)]\right)\right\}^{\alpha_2^N}$$

$$= \exp\left\{\alpha_2^N \log\left(1 + \sum_{j=1}^{p} [\exp(-r_j) - 1]\, \mathbb{1}_{\Delta_j^1}(\varepsilon)\, \mathbb{E}_\eta(\mathbb{1}_{\Delta_j^2}(\eta))\right)\right\} \tag{3.50}$$

Repeating what we did for (3.38) and (3.42), we get

$$(3.50) = \exp\left\{\sum_{j=1}^{p} [\exp(-r_j) - 1]\, \mathbb{1}_{\Delta_j^1}(\varepsilon)\, \alpha_2^N \mathbb{E}_\eta(\mathbb{1}_{\Delta_j^2}(\eta)\right\}\left(1 + \frac{C_N}{\alpha_2^N}\right) \qquad (3.51)$$

where C_N is a bounded function of S_i, t_i, r_i, $i = 1,..., p$. Therefore if

$$\hat{r}_j^{(N)} = (e^{-r_j} - 1)\, \alpha_2^N \mathbb{E}_\eta(\mathbb{1}_{\Delta_j^1}(\varepsilon))\left(1 + \frac{C_N}{\alpha_2^N}\right)$$

we get

$$\begin{aligned}(3.50) &= \prod_{j=1}^{p} \exp[\hat{r}_j^{(N)} \mathbb{1}_{\Delta_j^1}(\varepsilon)] \\ &= \prod_{j=1}^{p} \{1 + [\exp(\hat{r}_j^{(N)}) - 1]\, \mathbb{1}_{\Delta_j^1}(\varepsilon)\} \qquad (3.52)\end{aligned}$$

Since Δ_j^1 and Δ_k^1 are disjoint if $j \neq k$, expanding the product, we get

$$(3.52) = 1 + \sum_{j=1}^{p} [\exp(\hat{r}_j^{(N)}) - 1]\, \mathbb{1}_{\Delta_j^1}(\varepsilon)$$

Therefore

$$\begin{aligned}(3.49) &= \left\{1 + \sum_{j=1}^{p} [\exp(\hat{r}_j^{(N)}) - 1]\, \mathbb{E}_\varepsilon(\mathbb{1}_{\Delta_j^1}(\varepsilon))\right\}^{\alpha_1^N} \\ &= \exp \alpha_1^N \log\left\{1 + \sum_{j=1}^{p} [\exp(\hat{r}_j^{(N)}) - 1]\, \mathbb{E}_\varepsilon(\mathbb{1}_{\Delta_j^1}(\varepsilon))\right\} \qquad (3.53)\end{aligned}$$

Using (3.28), it is not difficult to check that

$$(3.53) = \exp \sum_{j=1}^{p} (\exp \hat{r}_j^{(N)} - 1)\, \alpha_1^N \mathbb{E}_\varepsilon(\mathbb{1}_{\Delta_j^1}(\varepsilon)(1 + \alpha_1^{-N} B_N)) \qquad (3.54)$$

where B_N is a bounded function of $\hat{r}_j^N$. Since

$$\lim_{N \to \infty} \hat{r}_j^{(N)} = (e^{-r_j} - 1)(e^{-s_j} - e^{-s_{j+1}})$$

and

$$\lim_{N \to \infty} \alpha_1^N \mathbb{E}_\varepsilon(\mathbb{1}_{\Delta_j^1}(\varepsilon)) = e^{-t_j} - e^{-t_{j+1}}$$

we get

$$\lim_{N\to\infty} \mathbb{E}\left(\exp\left[-\sum_{j=1}^{p} r_j \mathcal{N}_{\Delta_j}^{(N)}\right]\right)$$

$$= \prod_{j=1}^{p} \exp[(\exp - t_j - \exp - t_{j+1})$$

$$\times (\exp\{[\exp(-r_j) - 1](\exp - S_j - \exp - S_{j+1})\} - 1)]$$

$$= \prod_{j=1}^{p} \lim_{N\to\infty} \mathbb{E}(\exp[-r_j \mathcal{N}_{\Delta_j}^{(N)}]) \tag{3.55}$$

and this ends the proof of Theorem 3.

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